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FUNDAMENTAL SOLUTIONS FOR A FLUID-SATURATED POROUS SOLID

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Abstract—Inhomogeneous forcing functions, in the governing equations for linearized coupled deformation and pore-fluid diffusion, are characterized directly and *via* reciprocity theorem; inelastic straining or porosity changes, body-forces on fluid or solid and arbitrary fluid injection all elicit a response composed by appropriate distribution of point-force and fluid-source densities. The complete set of fundamental influence functions is established (*via* symmetry arguments) for an isotropic medium, so transparently that basic solutions for dipoles and point plasticity (slip or dilation) follow simply: the simulation of arbitrary anomalous zones of inelasticity is made rigorous in the process of proving dipole-equivalency for plasticity. Analytical and numerical implementation, for simulation of fracture phenomena in fluid-saturated porous media, is emphasized.

INTRODUCTION

There is a broad class of phenomena in soil and rock mechanics (e.g. as summarized by the author in Chap. 1 of [1]), of which the dominant characteristics are well represented by a model of localized plastic flow (and/or fluid injection) in an otherwise elastic medium composed of a deformation-controlling solid matrix with saturated interstices (viz. pore-space). Considerable study has been devoted to constitutive relations for the inelastic deformation (e.g. see the conference proceedings in [2]), particularly to the conditions for frictional slip and the inclusion of pore-fluid effects *via* classical effective stress laws for failure (e.g. Chap. 3 of [3]). However, the "consolidation" response of the saturated elastic medium (outside the zone of confined inelasticity), as a result of local slip, cracking or dilation, has not been analyzed: indeed, very few basic solutions of any kind are available for the coupled deformation-diffusion response that such a medium exhibits (e.g. see summary in Chap. 3 of [1]). The purpose of this paper is to provide a methodology and (for a linear isotropic model) a complete set of fundamental solutions (in the classical sense, e.g. [4]) which can, either directly or through numerical implementation, be applied to model quite general problems of consolidation and inelastic deformation (e.g. Chap. 4 of [1]) especially if "yielding" is localized in a fluid-saturated porous material.

To allow superposition of fundamental solutions the elasticity assumptions used will be linearized and the deformations are to be small (or incremental with suitable homogeneity restrictions): the model is a rationalized version of the classical Biot relations [5, 6], requiring only matrix strains (derivable from a solid displacement) and an internal variable called pore-pressure for complete specification of the isothermal quasi-static state. The parameters of the model are the drained (zero pore-pressure) and undrained (zero change in fluid content per unit volume) elastic moduli, together with classical coefficients (of soil mechanics) measuring the induced pore-pressure for undrained stressing. Only the content of untrapped fluid is variable and the porosity is the apparent volume fraction occupied by this freely filtrating fluid. The isotropic version (of most importance in this paper) has been rationalized by Rice and Cleary [7] and the anisotropic moduli have recently been similarly treated by the author. The resulting field equations are identical to those of linear *coupled* thermoelasticity (as noted also in [1, 7–9]) so one might expect an abundance of basic well-known solutions. Unfortunately, most solutions in thermoelasticity have been worked out after uncoupling the deformation from the diffusion equation (e.g. see review by Boley [10]), a procedure that is quite justifiable for typical values of thermoelastic constants (e.g. [11]) but which would completely lose the essence of porous media consolidation (again, see Chaps. 2 and 3 of [1] for rationalization of the thermoelastic analogy).

Indeed, very few worthwhile attempts have been made to solve the coupled, even isotropic, equations (an exception being Nowacki[12, 13] in the thermoelastic context), and many of these were specific to the poroelasticity applications. Biot provided some early solutions[14, 15] and then gave a displacement potential method[16] which has some advantage for particular techniques (e.g. that of Vargas[17], who employs Fourier transforms to solve the Biot equations for dynamic response to suddenly introduced point forces, an application of the pore-pressure theory which is often inappropriate[1]). Others have used transform techniques, both for dynamic[13] and quasi-static[18] problems (and, indeed, we could use such methods to find the solutions in this paper, but we consider that the procedures merely obfuscate simplifying features); their results are rarely rendered sufficiently transparent (or even tractable) for simple interpretation. A review may be found in Chap. 3 of [1] but here we shall take the most direct approach, which also happens to reveal characteristics of solutions for anisotropic or non-linear response (by keeping direct contact with the real region and forcing functions).

We begin by identifying the forcing functions in the governing equations, due to body forces on solid or fluid, injection of fluid or the occurrence of "plastic" deformation. A reciprocity theorem suggests, and condensation of field equations confirms, that the response to any of these multiple inhomogeneous terms can be constructed from a suitable distribution of point forces and fluid sources: thus, the minimum number of fundamental solutions for the linear theory is four for general anisotropy and two for an isotropic medium. The latter two solutions are then provided in such a transparent fashion that response to dipoles (shown to represent point plasticity by an Eshelby-type procedure) follows simply and we have all of the basic solutions we need for immediate application to localized inelastic deformation (e.g. anomalous dilatancy and faulting in the geophysical context).

CONSTITUTIVE EQUATIONS FOR A SATURATED POROUS MEDIUM

The material is composed of a solid matrix which contains interstitial pore-space filled with a freely diffusing pore-fluid. The mass of free fluid per unit volume is denoted by m which has an associated apparent volume fraction v defined by $m \equiv \rho v$, where ρ is the fluid density. Any trapped fluid (e.g. in adsorbed layers or non-communicating pore-space) is regarded as contributing to the net constitutive behaviour of the solid component, i.e. the moduli of the solid are to be regarded as effective moduli and the volume fraction of pore-space (v) is really an apparent fraction occupied by the free fluid. The deformation, when small or incremental, may be described by a suitable average vector displacement of the solid constituent, having components u_i in cartesian co-ordinates and giving rise to average cartesian strains ϵ_{ij}, defined by

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1}$$

Partial differentiation is denoted by a comma followed by the co-ordinate, $u_{i,j} \equiv \partial u_i/\partial x_j$ and we are using the usual Einstein notation. Changes in the volume fraction, Δv, and incremental strains ϵ_{ij}, will occur in response to changes (assumed quasi-static) in the dynamic variables which are chosen here as the *total* stresses σ_{ij} and the internal pressure on the pore-fluid p, assumed to have a local equilibrated value if deformation is sufficiently slow.

There is an elasticity associated with deformation of this porous medium; it is a result of the presumed existence of an energy function of the classical reversible thermostatic kind, i.e. we postulate that so-called "elastic" closed cycles of stressing may be performed in which the value of a suitable state function $U(\sigma_{ij}, p)$ also undergoes a closed cycle. A convenient choice for U, as a starting point, is the function whose perfect differential is given, when deformation is elastic, by

$$\mathrm{d}U = \sigma_{ij}\,\mathrm{d}\epsilon_{ij} + p\,\mathrm{d}v + T\,\mathrm{d}S \tag{2a}$$

where T is the local temperature and S is entropy per unit volume; this is a function U very similar to the internal energy per unit volume (Chap. 2 of [1]). Our concern will be with *isothermal* quasi-static deformation so it is appropriate to perform a Legandre transform to a free energy function $\Phi(\sigma_{ij}, p)$,

$$\Phi = U - \sigma_{ij}\epsilon_{ij} - p\Delta v - ST, \quad d\Phi = -\epsilon_{ij}\, d\sigma_{ij} - \Delta v\, dp. \tag{2b}$$

Clearly, then, elastic deformation measures are expressible as partial derivatives of $\Phi(\sigma_{ij}, p)$; this observation provides a means of identifying the "plastic" part $(\epsilon_{ij}^p, \Delta v^p)$ of the deformation, which we define (in a manner that seems arbitrary but is made rigorous in Chap. 2 of [1]) as the difference between the total deformation and its elastic part,

$$\epsilon_{ij}^p \equiv \epsilon_{ij} + \frac{\partial \Phi}{\partial \sigma_{ij}}, \quad \Delta v^p \equiv \Delta v + \frac{\partial \Phi}{\partial p}. \tag{2c}$$

Since $\Phi(\sigma_{ij}, p)$ is a state function, so are $\partial\Phi/\partial\sigma_{ij}$ and $\partial\Phi/\partial p$; like Biot[6] we linearize these elastic constitutive laws to obtain, in deference to symmetry implied by second derivatives of eqn (2c),

$$\epsilon_{ij} - \epsilon_{ij}^p = C_{ijkl}\sigma_{kl} + B_{ij}p, \quad \sigma_{ij} = \mathscr{L}_{ijkl}[(\epsilon_{kl} - \epsilon_{kl}^p) - B_{kl}p] \tag{3}$$

$$\Delta v - \Delta v^p = B_{mn}\sigma_{mn} + Dp = B_{mn}\mathscr{L}_{mnkl}(\epsilon_{kl} - \epsilon_{kl}^p) + (D - B_{mn}\mathscr{L}_{mnkl}B_{kl})p.$$

The compliance and stiffness tensors, C_{ijkl} and $\mathscr{L}_{ijkl}$, have the usual elastic symmetry, containing a maximum of 21 independent components when no material symmetry is present; B_{ij} is necessarily symmetric like ϵ_{ij}. We say nothing here about the constitutive equations for ϵ_{ij}^p and Δv^p, which are discussed at length elsewhere (e.g. Chap. 2 of [1]).

We need some constitutive assumptions about the pore-fluid: it is considered to be barotropic and its pressure-density curve is linearised. Thus, if the mass of free fluid per unit volume is $m = \rho v$, the change in m is approximated by

$$\Delta m = \rho_0 \Delta v + v_0 \Delta \rho = \rho_0(\Delta v + v_0 p/K_f) \tag{4}$$

where ρ_0, v_0 are the reference values of ρ, v (before applying σ_{ij} and p), and K_f is the bulk modulus of the fluid. Our last assumption concerns the rate at which the fluid transports under a potential gradient. This rate equation has also been investigated more carefully elsewhere (e.g. Chap. 2 of [1]) so we merely record the special linear *D'Arcy* version,

$$q_i = -\rho_0\kappa_{ij}[p_{,j} - \rho_0 f_j^F]. \tag{5}$$

Here q_i is the component of the fluid mass flow rate in the x_i direction, f_j^F are the components of the perturbation in the field of body-force per unit mass acting on the fluid and κ_{ij} are the components of the permeability tensor, usually given as k_{ij}/μ where μ is the dynamic viscosity of the fluid and k_{ij} have dimensions of area. The second order cartesian tensor κ_{ij} can be shown to be symmetric by familiar *Onsager* arguments (e.g. see [19]).

It is worth emphasizing here that the assumption of a unique ("locally equilibrated") pore-pressure p is a severe restriction which limits application of the theory, for most representative microstructures, to quasi-static deformation (e.g. see Chap. 2 of [1]). There have been many theories for dynamic response of porous media (e.g. see [17]) which implicitly retain the assumption of unique p; their conclusions (e.g. the dynamic counterpart of the point slip solution developed later), presumed mathematically correct, should be applied only to the very limited class of media for which the local equilibration time is sufficiently short[1].

CONSERVATION EQUATIONS

There are just two non-trivial conservation laws in the present isothermal quasi-static context: the first is a momentum balance or equilibrium equation

$$\sigma_{kl,l} + f_k = 0 \tag{6}$$

in which f_k is the change in body-force per unit volume (including that on solid and fluid). The second equation is that of fluid-mass conservation,

$$q_{k,k} + \frac{\partial m}{\partial t} = r^F \tag{7}$$

in which r^F is the rate of fluid-mass supply per unit volume and repeated subscripts denote summation. It will be convenient (as in [16]), to have a time-integrated version of eqn (7) so we introduce a total fluid displacement vector Q_i and a total volume of supplied fluid R^F to get

$$Q_{i,i} + \Delta m/\rho_0 = R^F; \quad q_i = \rho_0 \frac{\partial Q_i}{\partial t}, \quad r^F = \rho_0 \frac{\partial R^F}{\partial t}. \tag{8}$$

The array of governing eqns (1–8) may be combined to get a set of field equations; for instance, eqns (6–7) lead directly to four coupled equations in the unknowns u_i, p. These equations contain a variety of non-homogeneous terms; body-forces f_k^F, f_k, plastic deformation ϵ_{ij}^p, Δv^p and source-terms r^F all appear (as shown in Appendix), and we might fear that each inhomogeneous term calls for a separate fundamental Green function from which any field distribution of the inhomogeneity could be obtained by superposition; this would imply that 14 fundamental solutions are needed for the most general anisotropic body, reducing to 6 in the isotropic limit.

Actually, closer scrutiny of the field equations (Appendix) reveals that there is a strong correspondence in the status of the various inhomogeneous terms: for instance, the gradient of the plastic strain distribution is found to appear in the same context as the body forces f_k. The result of this correspondence will be that only 4 Green's functions are needed for the general anisotropic case and this number reduces to *a twain* in the isotropic limit. A convenient way to identify the correspondence, without resorting to field equations in particular variables, is by examining their status in a reciprocal relation between any two arbitrary solutions of the governing eqns (1)–(8). The appropriate reciprocity theorem is given next. The theorem actually provides an important means of establishing a range of useful solutions from knowledge of a few elementary solutions for the geometry in question; the simplest example would be the derivation of the solution for a slipped region in the body from that for a point-force.† However, we find an alternative way for establishing the inelastic solutions being sought in this work so only the theorem's ability to characterise ϵ_{ij}^p, Δv^p etc. will be emphasised in the present paper.

An even more important role of the reciprocal theorem has been defined by the author (Chap. 4 of [1]), namely as the basis of a boundary-integral equation scheme for solving eqns (1)–(8) in homogeneous regions of arbitrary geometries with boundary conditions on u_i (or $\sigma_{ij}n_i$, where n_i is normal to surface) and on p (or on $q_k n_k$). Since basic solutions of these coupled deformation and diffusion field equations are quite difficult to find analytically, except those derived later and some other special geometries, such numerical schemes are considered essential (and more economic, when applicable, than other, e.g. finite element, methods). Thus, for its twin value, the theorem is recorded here in isolation.

RECIPROCITY THEOREM FOR AN ELASTIC POROUS MEDIUM

We consider two separate "deformation" fields, $\{u_k^{(1)}, Q_k^{(1)}\}$ and $\{u_k^{(2)}, Q_k^{(2)}\}$ which are related to the "stress" fields $\{\sigma_{ij}^{(1)}, p^{(1)}\}$ and $\{\sigma_{ij}^{(2)}, p^{(2)}\}$ in the sense that they satisfy all of eqns (1)–(8) in the presence of the "forcing" fields $\{\epsilon_{ij}^{p(1)}, \Delta v^{p(1)}, f_k^{F(1)}, f_k^{(1)}, R^{F(1)}\}$ and $\{\epsilon_{ij}^{p(2)}, \Delta v^{p(2)}, f_k^{F(2)}, f_k^{(2)}, R^{F(2)}\}$. We prove (in Appendix) that this pair of otherwise completely arbitrary fields bear a reciprocal relation to each other in the following sense: for any arbitrary fixed volume V in space, surrounded by a surface S, the convolution products of the deformation with the stress

†Such an application of the reciprocal theorem (in the elasticity context, e.g. $p = 0$) has been shown by Rice and Chinnery [20] and is an alternative method, to the Eshelby-type procedure described later, for arriving at the force-dipole equivalency of point slip or extension.

variables satisfy the equation

$$\int_S \{\tilde{T}_k^{(1)}\tilde{u}_k^{(2)} - \tilde{T}_k^{(2)}\tilde{u}_k^{(1)}) - (\tilde{p}^{(1)}\tilde{Q}_n^{(2)} - \tilde{p}^{(2)}\tilde{Q}_n^{(1)})\}\,\mathrm{d}S + \int_V \{(\tilde{F}_k^{(1)}\tilde{u}_k^{(2)} - \tilde{F}_k^{(2)}\tilde{u}_k^{(1)} + \rho_0\tilde{f}_k^{F(1)}\tilde{Q}_k^{(2)}$$

$$- \rho_0\tilde{f}_k^{F(2)}\tilde{Q}_k^{(1)})\}\,\mathrm{d}V + \int_V \{\tilde{p}^{(1)}\tilde{\Gamma}^{F(2)} - \tilde{p}^{(2)}\tilde{\Gamma}^{F(1)}\}\,\mathrm{d}V = 0. \tag{9a}$$

in which we have introduced the following notation

$$T_k \equiv (\sigma_{kl} + \mathscr{L}_{ijkl}\epsilon_{ij}^p)n_l \ , \ Q_n \equiv Q_k n_k$$

$$F_k \equiv f_k - (\mathscr{L}_{ijkl}\epsilon_{ij}^p)_{,l} \tag{9b}$$

$$\Gamma^F \equiv R^F - \Delta v^p + \mathscr{L}_{ijkl}B_{kl}\epsilon_{ij}^p$$

and where a superposed tilde is used to denote the Laplace transform of the variable, for instance

$$\tilde{p}(x_i, s) \equiv \int_0^\infty p(x_i, t)e^{-st}\,\mathrm{d}t. \tag{9c}$$

We are using the Laplace transform to give a convenient representation of convolution products as products of Laplace transforms: if $h(t)$ is any function defined as the time convolution of two other functions, $f(t)$ and $g(t)$, namely

$$h(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau \tag{9d}$$

then the Laplace transform of h is $\tilde{h}(s) = \tilde{f}(s)\tilde{g}(s)$.

Each of the products in eqn (9a) is the transform of a time function resembling energy supplied to the material within volume V, but the resemblance is actually just formal. A particular example, when the field $\{u_k^{(1)}, Q_k^{(1)}, \sigma_{kl}^{(1)}, p^{(1)}\}$ is a real field solving a given boundary value problem and $\{u_k^{(2)}, Q_k^{(2)}, \sigma_{kl}^{(2)}, p^{(2)}\}$ is some arbitrary variation on that field satisfying eqns (1)–(8) and maintaining the given boundary conditions, leads to a useful variational principle for numerical analysis (e.g. Chap. 4 of [1]). $T_k u_k$ resembles work done by tractions on the boundary S and T_k contains an additional traction stress term $\mathscr{L}_{ijkl}\epsilon_{ij}^p$ which must be included in computing the energy supply $T_k u_k$. More interesting is the observation that the gradient of plastic strain $(\mathscr{L}_{ijkl}\epsilon_{ij}^p)_{,l}$ appears in the same role as the total body forces and that fundamental influence functions for localised point forces may therefore be used to simulate this particular effect of a plastic strain distribution.

The forcing function ϵ_{ij}^p also appears in the role of a fluid source density $\mathscr{L}_{ijkl}B_{kl}\epsilon_{ij}^p$, along with R^F and Δv^p; the latter has quite a simple physical interpretation as the fluid volume needed to fill the additional created pore-space but $\mathscr{L}_{ijkl}B_{kl}\epsilon_{ij}^p$ is somewhat more subtle: it is the volume of fluid (per unit total volume) which must be removed, in order to maintain zero pore-pressures, after a plastically deformed element has been restored to its initial state by imposing reverse total stress increments (see the last of eqns (3)). We shall discuss this (Eshelby-type) concept of restoring the original state of an element after it is plastically deformed, when we derive the influence functions for point slip and point dilation later on.

What has been recognised by writing eqn (9) is that the fundamental solutions for all of the forcing terms ϵ_{ij}^p, Δv^p, f_k^F, f_k, Γ^F may be derivable from those for f_k, r^F only i.e. from point-force and fluid source solutions only. What has not been identified by eqn (9) is that the forcing functions f_k, f_k^F, ϵ_{ij}^p also generate a distribution of fluid-source dipoles; loosely, we might attribute this failure to the fact that such dipoles do not have any associated net energy absorption. The dipoles do, however, appear in the governing field equation of diffusion. Since the next step in this work is to establish fundamental point force and source solutions for an isotropic medium, it will be convenient to set out all the governing field equations for such a medium (in the next section) and these will reinforce the correspondence among forcing functions just established; they will also reveal the dipole distribution.

FIELD EQUATIONS FOR AN ISOTROPIC MEDIUM

When the field equations are written for a general anisotropic porous solid (Appendix 1), much of the correspondence among forcing terms is exposed, just as it is by eqn (9). However, these most general equations are much less tractable and somewhat less revealing (especially in the resulting form of the diffusion equation) than their special isotropic limit. Although physical considerations suggest the necessity of an anisotropic model (e.g. geophysical applications), our philosophy here is that the isotropic model is by no means exhausted (e.g. as to identification of primary features of subterranean faulting and fracturing) and it provides useful understanding with which anisotropic solutions can be compared, as they become available. We have argued elsewhere (Chap. 1 of [1]) that the isotropic Biot model seems capable of describing many time-dependent aspects of rupture phenomena. The remainder of our analysis is therefore devoted to extracting basic solutions for an isotropic saturated porous medium.

The parameters and moduli of an isotropic medium have been discussed by Rice and Cleary[7] and, in the notation of eqn (3), are as follows:

$$C_{ijkl} = \left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2\nu}{(1+\nu)}\delta_{ij}\delta_{kl}\right]\Big/4G \tag{10a}$$

$$\mathcal{L}_{ijkl} = G\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \frac{2\nu}{(1-2\nu)}\delta_{ij}\delta_{kl}\right] \tag{10b}$$

$$B_{ij} = \frac{3(\nu_u - \nu)}{2GB(1+\nu)(1+\nu_u)}\delta_{ij}, \quad \kappa_{ij} = \kappa\delta_{ij} \tag{10c}$$

$$D = \frac{1}{B}\left(\frac{1}{K} - \frac{1}{K_s'}\right), \quad K \equiv \frac{2G(1+\nu)}{3(1-2\nu)}. \tag{10d}$$

The "drained" Poisson ratio of the solid matrix is ν, G is the shear modulus, ν_u is the "undrained" Poisson ratio and B is the induced pore-pressure parameter of Skempton[21]. By "undrained" we mean the response immediately after application of stress and its significance is made obvious by deriving a more transparent form of eqn (4), namely

$$\Delta m^e \equiv \Delta m - \rho_0 \Delta v^p = \frac{3\rho_0(\nu_u - \nu)}{2GB(1+\nu)(1+\nu_u)}\left[\sigma_{kk} + \frac{3}{B}p\right]. \tag{11}$$

By noting that $\Delta m^e = 0$ immediately upon application of a stress increment σ_{ij}, we observe that the instantaneous induced pore-pressure is $p_{\text{inst.}} = -B\sigma_{kk}/3$. The last parameter in eqns (10) is the effective bulk modulus of the non-mobile ("solid") constituent, K_s'.

Our present choice will be to combine eqns (1), (3), (6) to obtain a set of modified Navier equilibrium equations for displacement

$$G[u_{i,kk} + (1-2\nu)^{-1}u_{k,ki}] - \frac{3(\nu_u - \nu)}{B(1+\nu_u)(1-2\nu)}p_{,i} = 2G\left[\epsilon^p_{ik,k} + \frac{\nu}{1-2\nu}\epsilon^p_{kk,i}\right] - f_i \equiv -F_i \tag{12}$$

in which the gradients of ϵ^p_{ij} appear clearly in the role of body-forces. (Of course, $p_{,i}$ appears also in that role but cannot be regarded as a known distribution since its behaviour is coupled to the solution for $u_{k,k}$ as we shall see immediately). A very useful consequence of eqn (12) is obtained by computing its divergence and converting from $u_{k,k}$ to σ_{kk}:

$$\left[\sigma_{kk} + \frac{6(\nu_u - \nu)}{B(1-\nu)(1+\nu_u)}p\right]_{,ll} = -\left(\frac{1+\nu}{1-\nu}\right)[2G(\epsilon^p_{kk,ll} - \epsilon^p_{k,kl}) + f_{k,k}]. \tag{13}$$

We shall find that this is the only compatibility equation needed for the class of problems considered in this analysis. We could equally well have chosen to write compatibility equations on stress (like Rice and Cleary [7]) instead of eqns (12).

We can manipulate eqns (5), (7), (11) and (13) to obtain a diffusion equation of the form

$$c(\Delta m^e + \Gamma^e)_{,ll} - \frac{\partial}{\partial t}(\Delta m^e + \Gamma^e) = -\gamma^F \tag{14a}$$

where we have employed the abbreviation γ^F for a net fluid supply rate

$$\gamma^F \equiv r^F - \frac{\partial}{\partial t}(\rho_0 \Delta v^p - \Gamma^e) - \rho_0{}^2 \kappa f^F_{k,k} + \frac{\rho_0 \kappa B(1+\nu_u)}{3(1-\nu_u)} F_{k,k} \tag{14b}$$

and the other fluid source Γ^e derives from "elastic" restoration of plastic dilation,

$$\Gamma^e = \frac{3\rho_0(\nu_u - \nu)\epsilon^p_{kk}}{B(1-2\nu)(1+\nu_u)}. \tag{14c}$$

Actually, Γ^e is exactly the term $\rho_0 \mathscr{L}_{ijkl} B_{kl} \epsilon^p_{ij}$ mentioned in and after eqn (9) and its appearance in eqn (14a) is perfectly consistent with its interpretation as the change of fluid mass when a dilated element is restored to its pre-dilation state while allowing all pore-pressures to damp out to zero. The parameter of *diffusivity* c was deduced by Rice and Cleary [7] to be

$$c = \frac{2\kappa G B^2 (1+\nu_u)^2 (1-\nu)}{9(1-\nu_u)(\nu_u - \nu)}. \tag{14d}$$

It is worth inspecting eqn (14a) to see that it does contain all of the fluid source characteristics of ϵ^p_{ij} and Δv^p exactly as predicted by eqn (9) i.e. $r^F - \rho_0(\Delta v^p - \mathscr{L}_{ijkl} B_{kl} \epsilon^p_{ij})$ is the net source generated and sudden occurrences of Δv^p and ϵ^p_{ij} generate instantaneous sources. But eqn (14) also exposes the fact that both $f_k{}^F$ and F_k (eqn (12)) give rise to a source density which can very quickly be shown to have the character of a distribution of dipole pairs of sources and sinks; this is recognised, for instance, by considering a single localised (say bell-shaped) distribution of $f_k{}^F$ or F_k which, in the limit of localisation, generates a single dipole: then any given field of $f_k{}^F$ or F_k may be composed of a suitable density of the single point forces, leading (by superposition) to a density of dipole pairs. We must be careful to distinguish between the stress fields of $f_k{}^F$ and F_k, however, since the former generates a pure dipole while the latter must also be equilibrated by the stress field: the fluid flow field of $f_k{}^F$ reaches a steady state at $t = \infty$ while that of F_k (we shall see) damps out to extinction. These points will be more readily understood in the later section on solutions for sources and dipoles.

Equations (12) and (14) reinforce the assertion that the effects of all the forcing functions on both solid displacement and fluid flow may be reduced to those of an appropriate distribution of localised forces and fluid sources: thus any combination of ϵ^p_{ij}, Δv^p, $f_k{}^F$, f_k, r^F may be simulated once we have the solution for an arbitrarily oriented point force and a fluid source in the material and geometry of interest. These may not be easy to establish in general but they can be found for special cases: one such useful pair of solutions (for an infinite isotropic medium), and their application to deriving more directly usable Green's functions, are the object of the remainder of this analysis.

The solutions which we develop actually fit into either of two classes of problems associated with eqns (12)–(14). The first class (containing all unidimensional problems, including the fluid source, for instance) is that for which u_k is the gradient of a scalar ϕ, $u_k = \phi_{,k}$; the implications for eqns (12) are

$$\phi_{,kk} = \epsilon_{kk} = \frac{3(\nu_u - \nu)}{2GB(1-\nu)(1+\nu_u)} p + \left(\frac{\nu}{1-\nu}\right)\epsilon^p_{kk} + g \tag{15a}$$

where we have introduced a (perhaps artificial) scalar potential g which may (or may not) have an actual physical meaning as the body-force potential

$$g_{,i} = \left(\frac{1-2\nu}{1-\nu}\right)[\epsilon^{p}_{ik,k} - f_i/2G]. \tag{15b}$$

The consequence of eqn (15a) is that eqn (14a) may be reduced to a diffusion equation in p alone: it is notationally convenient to write this equation in terms of a variable p^e, analogous to $\Delta m^e + \Gamma^e$,

$$p^e_{,ll} - \frac{1}{c}\frac{\partial p^e}{\partial t} = -\frac{\gamma^F}{\rho_0 \kappa}, \quad p^e \equiv p + \frac{2GB(1+\nu_u)}{3(1-\nu_u)(1-2\nu)}[\epsilon^p_{kk} + (1-\nu)g]. \tag{15c}$$

If all forcing functions are absent then only g survives and it is spatially constant (eqn 15b) so that eqn (15c) becomes

$$cp_{,ll} - \frac{\partial p}{\partial t} = \frac{2GB(1+\nu_u)(1-\nu)}{3(1-\nu_u)(1-2\nu)}\frac{\mathrm{d}g(t)}{\mathrm{d}t} \tag{15d}$$

and even the latter inhomogeneity vanishes if the region is infinite; however, g can have great importance in bounded regions (e.g. the annular specimen in [7]).

The second class of problems is that for which the "harmonic" function in eqn (13) is time-independent,

$$\frac{\partial}{\partial t}\left[\sigma_{kk} + \frac{6(\nu_u - \nu)}{B(1-\nu_u)(1+\nu_u)}p\right] = 0 \tag{16}$$

and this class will be shown here to contain all problems of embedded body-forces and "plasticity" in an infinite region. Rice and Cleary[7] noticed the plane-strain version of the time-independent function in eqn (16) when they solved for the field of long straight edge dislocations or lines of concentrated body force; since we find it to apply for the point force solution considered next, and then prove that all embedded slip or cracking is derivable as a distributed density of such singular solutions, we conclude that all surface discontinuity problems in an infinite medium have the characteristic of eqn (16), which greatly simplifies their solution.

POINT FORCE IN AN INFINITE MEDIUM

The first fundamental solution which we seek is that for a forcing function of the kind

$$f_k = P_k H(t)\delta(x_1)\delta(x_2)\delta(x_3) \tag{17}$$

where $\delta(x)$ and $H(t)$ are the usual Dirac delta and Heaviside step functions. It transpires that our solution will automatically display the fluid source dipole effect suggested by eqn (14) if we enforce the inhomogeneity which f_k causes in eqns (3), (6) and (12). We set all other forcing terms (f_k^F, r^F, Δv^p) to zero, merely noting the obvious duality of f_k and $(\mathscr{L}_{ijkl}\epsilon^p_{ij})_{,l}$ in eqn (12) which is explored later in detail.

The labour of solution is minimised by noting the high degree of symmetry connected with the field of the point force in eqn (17); in Fig. 1, we show the force acting in the x_3-direction, without loss of generality for an isotropic material, and it is apparent immediately that the problem is axisymmetric with respect to the axis x_3. In terms of the spherical co-ordinates superposed on the diagram, the field of influence is independent of the meridional angle ϕ and the meridional displacement u_ϕ vanishes so both of the shear-stresses $\sigma_{r\phi}$, $\sigma_{\theta\phi}$ vanish and $\sigma_{\phi\phi}$ may be deduced once the solution in the (r, ϕ)-plane is known.

The result of these observations is that only eqns (6), (13) and (14) will be needed to obtain the stress-field of the point-force (and thence the displacement field). The procedure developed below is considered to be much more revealing than that which results from application of conventional transform techniques (e.g. those of McNamee and Gibson[18] and Nowacki[13]).

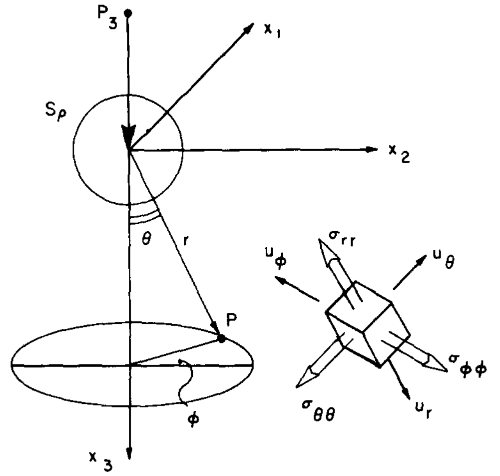

Fig. 1. Point force at the origin of an infinite medium; spherical coordinates are attached and axisymmetry of the response in an isotropic medium is implied.

A corollary of our solution is that of Kelvin for a point force in an infinite elastic medium.

We begin by noting that pore-pressure p is a scalar variable which enters as the parameter controlling the variation of an elastic stress field; its evolution is governed by the diffusion eqn (14) but there is no characteristic length present in the problem except that provided by the diffusion length $\sqrt{ct}$. All equations are linear so p must be linear in the arbitrarily oriented P_k and it must have dimensions of stress: we can find only one form which respects all of these considerations,

$$P = \frac{P_k x_k}{r^3} f_1(\xi), \quad \xi \equiv r/\sqrt{ct} \tag{18}$$

where f_1 is a scalar function, which may contain the material constants.

It is advantageous, especially as a first step toward investigating basic solutions in anisotropic (or nonlinear) media, to put our line of reasoning in the context of general restrictions which arise from material symmetry. These considerations have led to so-called canonical representation theorems (e.g. see Wineman and Pipkin [22]) for the dependence of a response tensor on a set of tensors containing forcing, position and perhaps other tensors: thus, they are useful both in formulation of constitutive relations and also in limiting the possible forms of functions describing the field of influence of discrete forcing tensors once the constitutive behaviour has been decided. Ours is almost the simplest possible example for use of these theorems: the response tensor is the stress field, the forcing tensor is adequately described by the vector P_k and the other independent variable is the vector of position x_k.

It is even more convenient to start with the displacement response u_k to P_k at x_k because this permits us to give a simple demonstration of the kind of logic that leads, in the more general case, to the theorems mentioned above. One first examines the symmetry of the problem: Fig. 2 shows the transformation invariance implied by isotropy, namely that a rotation of P_k and x_k by some (proper or improper) orthogonal transformation leads to a rotation of $u_k(P_i, x_j)$ by exactly the same transformation. One then introduces an auxiliary vector a_k to form the system a_k, P_k, x_k; the next, most difficult, step is to establish the independent elements, computed as polynomials of the components of a_k, P_k, x_k, which are invariant under the transformation of this system. The third step is to single out those elements I_α ($\alpha = 1, \ldots, A$) which are functions of P_k, x_k only (a so-called "integrity" basis for invariants of this pair of vectors under the transformation): these are P_kP_k, P_kx_k and x_kx_k for present purposes. Next, one selects those invariants J_β of the system a_k, P_k, x_k which are linear in a_k because, for instance, we observe that a_ku_k is an invariant of the transformation: there are just two distinct kinds of J_β here, namely

$$J_1 = a_k x_k f^{(2)}(P_kP_k, P_kx_k, x_kx_k), \quad J_2 = a_k P_k f^{(3)}(P_kP_k, P_kx_k, x_kx_k) \tag{19a}$$

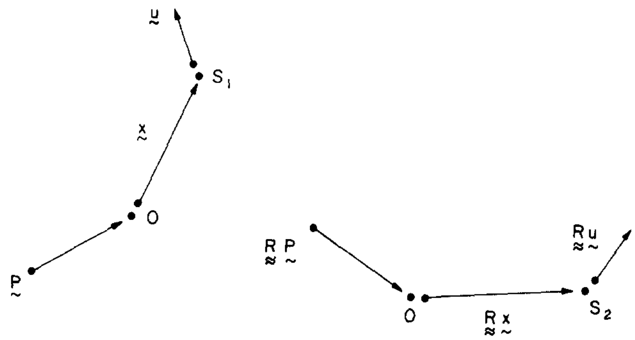

Fig. 2. Symmetry of the response to a point force in an infinite isotropic medium; the triplicate of point force, position and resulting displacement vectors displays a "rigid body" rotation under an orthogonal transformation.

where $f^{(2)}$, $f^{(3)}$ are (usually tensor) polynomials of arbitrary order and $P_k f^{(3)}$, $x_k f^{(2)}$ are the so-called "basic form-invariant tensors" of the present problem. The canonical representation theorem of Wineman and Pipkin[22] then states that

$$u_k = p_1(I_1, \ldots, I_A) f^{(2)}(I_1, \ldots, I_A) x_k + p_2(I_1, \ldots, I_A) f^{(3)}(I_1, \ldots, I_A) P_k \tag{19b}$$

where p_1, p_2 are scalar polynomials and (for the system of Fig. 1) the independent variables in $f^{(2)}$, $f^{(3)}$ just happen to be scalar and identical to those in p_1, p_2.

Now we simply observed that all our eqns (12)–(14) are linear and that u_k is linear and homogeneous in P_k; u_k has dimensions of length so the only possible form it can have is

$$Gu_i = f_2(\xi)\left(\frac{x_k P_k}{r^3}\right)x_i + f_3(\xi)\frac{P_i}{r} \tag{19c}$$

because G is the only material parameter available to convert from force to length. The functions f_2, f_3 may be arbitrary polynomials in $x_k x_k$ (but the latter has necessarily been made dimensionless by the diffusion length $\sqrt{ct}$) and may contain the other dimensionless material parameters.

By combining eqns (3), (10), (18) and (19) we can establish the inevitable structure of the stress-field

$$\sigma_{ij} = \frac{x_i x_j}{r^2}\left[\frac{x_k P_k}{r^3} F_1(\xi)\right] + \frac{x_i P_j + x_j P_i}{r^3} F_2(\xi) + \delta_{ij}\left[\frac{x_k P_k}{r^3} F_3(\xi)\right] \tag{20a}$$

and these new functions F_1, F_2, F_3 incorporate the displacement and pore-pressure evolution functions:

$$\begin{aligned} F_1(\xi) &\equiv 2\xi^4 \frac{\mathrm{d}}{\mathrm{d}\xi}[\xi^{-3} f_2], \quad F_2(\xi) \equiv f_2 + \xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^{-1} f_3) \\ F_3(\xi) &\equiv \frac{2}{(1-2\nu)}\left[(1-\nu) f_2 - \nu f_3 + \nu\xi(f_2' + f_3') - \frac{3(\nu_u - \nu)}{2B(1+\nu_u)} f_1\right]. \end{aligned} \tag{20b}$$

We have introduced the ordinary differentiation notation $f'(\xi) \equiv \mathrm{d}f(\xi)/\mathrm{d}\xi$ in eqn (20b) and shall employ it hereafter. To complete the solution for the stress field (eqn 20a) and the pore-pressure field (eqn 18) it remains to determine the four evolution functions F_1, F_2, F_3, f_1.

DETERMINATION OF EVOLUTION FUNCTIONS

We shall not need to use eqns (12) in the form given because eqns (6), (13) and (14) will allow us to determine the stress-field to within an arbitrary multiplicative constant which can then be deduced by using the relations in eqn (20b). The procedure is to apply eqns (6), of which only two are independent because of axisymmetry, then to solve eqn (14) which produces the unknown constant, and lastly to employ eqn (13). Substitution of eqn (20a) into eqns (6) yields

$$[F_2+F_3+\xi F_2']r^2P_i+[\xi F_1'-3(F_2+F_3)+\xi(F_2'+F_3')](x_kP_k)x_i=0 \tag{21}$$

from which we obtain two independent conditions

$$F_2+F_3+\xi F_2'(\xi)=0,\quad \xi\frac{\mathrm{d}}{\mathrm{d}\xi}[F_1+4F_2+F_3]=0. \tag{22a}$$

Actually, the second of eqns (22a) is the same as we would derive by imposing the condition (Fig. 1) that the stress-field equilibrates a point-force: we integrate eqns (6) over any sphere with a finite radius ρ, enclosing the point of force application, and then employ the divergence theorem to get

$$-P_i=\int_{r\leq\rho}\sigma_{ij,j}\,\mathrm{d}v=\int_{r=\rho}\sigma_{ij}\left(\frac{x_j}{r}\right)\mathrm{d}S \tag{22b}$$

However, by noting that the surface integral of x_ix_k is zero unless $i=k$, we obtain additionally the *integrated* form of $(22a)_2$, namely

$$F_1+4F_2+F_3=-\frac{3}{4\pi}. \tag{22c}$$

In using eqn (14) we employ the abbreviation

$$\Delta m^e=\frac{3\rho_0(\nu_u-\nu)x_kP_k}{2GB(1+\nu)(1+\nu_u)r^3}F(\xi),\quad F=F_1+2F_2+3F_3+\frac{3}{B}f_1 \tag{23}$$

and thus derive the ordinary differential equation

$$F''(\xi)-\left(\frac{2}{\xi}-\frac{\xi}{2}\right)F'(\xi)=0. \tag{24}$$

When we impose the condition that $\Delta m^e=0$ when $t=0$, the solution of eqn (24) is

$$F=\frac{F_\infty}{2\sqrt{\pi}}\int_\xi^\infty \eta^2\mathrm{e}^{-\eta^2/4}\,\mathrm{d}\eta \tag{25}$$

where F_∞ is the unknown value of F at $t=\infty$ which will be determined only at the very end of this analysis. A last relation between the evolution functions is very simply determined by applying eqn (13) at all non-singular points; again we abbreviate to

$$\sigma_{kk}+\frac{6(\nu_u-\nu)}{B(1-\nu)(1+\nu_u)}p=\left(\frac{x_kP_k}{r^3}\right)G(\xi),\quad G=F-\frac{3}{B}\left[\frac{(1+\nu)(1-\nu_u)}{(1-\nu)(1+\nu_u)}\right]f_1 \tag{26}$$

and the resulting ordinary differential equation is

$$G''(\xi)-\frac{2}{\xi}G'(\xi)=0 \tag{26a}$$

which has a simple homogeneous solution with constants readily determined by the conditions of vanishing stresses at infinite distance from the point-force and $f_1=0$ at $t=\infty$, namely†

$$G(\xi)=k_1\xi^3+k_2;\quad k_1\equiv 0,\ k_2=F_\infty. \tag{26b}$$

†Note that here lies the proof of eqn (16) for the fundamental point-force solution.

The collection of equations for F_1, F_2, F_3 now simplifies to

$$F_2 - F_3 = \xi^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^2 F_2), \quad F_1 + 4F_2 + F_3 = -\frac{3}{4\pi} \tag{27}$$

$$F_1 + 2F_2 + 3F_3 = \frac{F_\infty}{(1+\nu)(1-\nu_u)}\left[(1-\nu)(1+\nu_u) - \frac{2(\nu_u-\nu)}{2\sqrt{\pi}}\int_\xi^\infty \eta^2 \mathrm{e}^{-\eta^2/4}\,\mathrm{d}\eta\right]$$

which equations have the solution

$$\begin{aligned} F_2 &= -\frac{1}{4}\left[\frac{3}{4\pi} + F_\infty\right] - \frac{\omega}{2\sqrt{\pi}}\int_0^\xi (1-\eta^2/\xi)\eta^2 \mathrm{e}^{-\eta^2/4}\,\mathrm{d}\eta \\ F_3 &= \frac{1}{4}\left[\frac{3}{4\pi} + F_\infty\right] + \frac{\omega}{2\sqrt{\pi}}\int_0^\xi (1+\eta^2/\xi^2)\eta^2 \mathrm{e}^{-\eta^2/4}\,\mathrm{d}\eta \\ F_1 &= -\frac{3}{4}\left[\frac{1}{4\pi} - F_\infty\right] + \frac{\omega}{2\sqrt{\pi}}\int_0^\xi (3-5\eta^2/\xi^2)\eta^2 \mathrm{e}^{-\eta^2/4}\,\mathrm{d}\eta. \end{aligned} \tag{28}$$

Here we have used the temporary notation $\omega \equiv 2F_\infty(\nu_u-\nu)/[4(1+\nu)(1-\nu_u)]$.

The first pair of eqns (20b) now provide the solution

$$\begin{aligned} f_2 &= \frac{1}{8}\left(\frac{1}{4\pi} - F_\infty\right) - \frac{\omega}{4\sqrt{\pi}}\int_0^\xi \left(1-\frac{\eta^2}{\xi^2}\right)\eta^2 \mathrm{e}^{-\eta^2/4}\,\mathrm{d}\eta \\ f_3 &= \frac{1}{8}\left(\frac{7}{4\pi} + F_\infty\right) + \frac{\omega}{4\sqrt{\pi}}\int_0^\xi \left(1-\frac{\eta^2}{3\xi^2}\right)\eta^2 \mathrm{e}^{-\eta^2/4}\,\mathrm{d}\eta + \frac{\omega\xi}{3\sqrt{\pi}}\mathrm{e}^{-\xi^2/4}. \end{aligned} \tag{29}$$

Finally, the last of eqns (20b) confirms our algebra and enables us to solve for F_∞,

$$F_\infty = \frac{-(1+\nu)}{4\pi(1-\nu)}. \tag{30}$$

It will be convenient to have a compact listing of the evolution functions:

$$8\pi(1-\nu)\begin{Bmatrix} -F_1 \\ -F_2 \\ F_3 \\ f_1 \\ 2f_2 \\ 2f_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ (1-2\nu) \\ (1-2\nu) \\ 0 \\ 1 \\ (3-4\nu) \\ -\frac{4\xi}{3}\left(\frac{\nu_u-\nu}{1-\nu_u}\right)\mathrm{e}^{-\xi^2/4} \end{Bmatrix} + \frac{(\nu_u-\nu)}{2\sqrt{\pi}(1-\nu_u)}\int_0^\xi \mathrm{d}\eta \begin{Bmatrix} 3-5\eta^2/\xi^2 \\ \eta^2/\xi^2-1 \\ -\eta^2/\xi^2-1 \\ \frac{2B(1+\nu_u)(1-\nu)}{3(\nu_u-\nu)} \\ 1-\eta^2/\xi^2 \\ \eta^2/3\xi^2-1 \end{Bmatrix}\eta^2\mathrm{e}^{-\eta^2/4} \tag{31}$$

Equations (20a) and the first three give the stress field, eqn (18) and f_1 give the pore-pressure and then eqn (19c) with the last two, gives the displacement field. Notice that f_1, f_2, F_1, F_2, F_3 simply evolve from constants (containing ν_u) at $t=0$ ($\xi=\infty$ for $r\neq 0$) to corresponding constants (except that ν_u is replaced by ν) at $t=\infty$ ($\xi=0$), while f_1 evolves from $B(1+\nu_u)/12\pi(1-\nu_u)$ at $t=0$ to zero at $t=\infty$. Thus, as expected, the field evolves from undrained (G, ν_u, $\Delta m^e=0$) to drained (G, ν, $p=0$) elastic response.

POINT SOURCES AND FLUID DIPOLES

The second fundamental singular solution needed to simulate the response to forcing functions Δv^P, ϵ_{ij}^P, f_k^F, f_k, r^F is that for the local injection of a specified amount of fluid, idealized as fluid supply at a point in order to obtain the Green function. In the notation of eqn (14a), we are considering a fluid supply rate of $f(t)$ and thus

$$\gamma^F = f(t)\delta(x_1)\delta(x_2)\delta(x_3). \tag{32}$$

The problem is necessarily spherically symmetric and thus fits into the class of solutions for which eqns (15) are valid; it is interesting to scrutinize the special spherically symmetric version of eqns (6), (13), (15c) so we list the equations here (without the elementary derivation given in [7] and in the notation of Fig. 1, with $\sigma_{\phi\phi} = \sigma_{\theta\theta}$). Partly, we wish to emphasize a feature causing error in previous treatments of geometries annular in cross-section (e.g. [9]), namely the presence of $g(r, t)$ as follows:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{2(\sigma_{rr} - \sigma_{\theta\theta})}{r} = -f_r \tag{33}$$

$$\sigma_{rr} + 2\sigma_{\theta\theta} + \frac{6(\nu_u - \nu)}{B(1+\nu_u)(1-\nu)}p = \frac{2G(1+\nu)}{(1-2\nu)}\left[g(r,t) - \left(\frac{1-2\nu}{1-\nu}\right)\epsilon^P_{kk}\right] \tag{34}$$

$$\frac{\partial^2 p^e}{\partial r^2} + \frac{2}{r}\frac{\partial p^e}{\partial r} = \frac{1}{c}\frac{\partial p^e}{\partial t} - \frac{\gamma^F}{\rho_0\kappa}. \tag{35}$$

These are the equations of Rice and Cleary [7] with the addition of forcing functions defined in eqns (2), (6), (15) and (14). The variable g is spatially constant if the forcing functions are zero and it vanishes if the region is infinite.

It will suffice to give the solutions to eqns (33)–(35) for two special choices of $f(t)$, from either of which the response to any general $f(t)$ may be computed by Stieltjes or Duhamel time integration. The first choice is $f(t) \equiv Q^{eff}\delta(t)$, namely the injection of a mass Q^{eff} of fluid instantaneously at $t = 0$ (we use Q^{eff} to connect with a latter use of this quantity to denote the effective strength of generated fluid sources). The solution to eqn (35) in an infinite region is then well known (but best rationalised in Boley and Weiner [11], p. 166) to have the form

$$p = \frac{Q^{eff}c}{4\pi\rho_0\kappa r^3}\frac{\xi^3 e^{-\xi^2/4}}{2\sqrt{\pi}}, \quad \xi \equiv r/\sqrt{ct}. \tag{36}$$

The forcing functions in eqns (33) and (34) are presently set to zero so integration of eqn (33), after eliminating $\sigma_{\theta\theta}$, can be shown to produce

$$\sigma_{rr} = \frac{\eta Q^{eff}c}{\rho_0\kappa\pi r^3}\left[\frac{\xi}{\sqrt{\pi}}e^{-\xi^2/4} - \mathrm{erf}\,(\xi/2)\right], \quad \eta \equiv \frac{3(\nu_u - \nu)}{2B(1+\nu_u)(1-\nu)} \tag{37a}$$

and then $\sigma_{\theta\theta}$ is trivially determined to be

$$\sigma_{\theta\theta} = \frac{\eta Q^{eff}c}{2\rho_0\kappa\pi r^3}\left[\mathrm{erf}\left(\frac{\xi}{2}\right) - \frac{\xi}{\sqrt{\pi}}\left(1 + \frac{\xi^2}{2}\right)e^{-\xi^2/4}\right] \tag{37b}$$

where

$$\mathrm{erf}\,(\xi) \equiv \frac{2}{\sqrt{\pi}}\int_0^\xi e^{-x^2}\,dx.$$

The second choice is that of $f(t) = qH(t)$, describing a constant rate (q) of local fluid supply which begins at $t = 0$. The solution of eqn (35) is again well known (e.g. see Carslaw and Jaegar [23]), namely

$$p = \frac{q}{4\pi\rho_0\kappa r}\left[1 - \mathrm{erf}\left(\frac{\xi}{2}\right)\right] \tag{38}$$

and from this, through eqns (33) and (34) as before, we can establish the stress field

$$\sigma_{rr} = \frac{-\eta q}{2\pi\rho_0\kappa r}\left\{\left(1-\operatorname{erf}\left(\frac{\xi}{2}\right)\right)+\frac{1}{\sqrt{\pi}}\int_0^{\xi}\frac{s^2}{\xi^2}e^{-s^2/4}\,ds\right\} \tag{39a}$$

$$\sigma_{\theta\theta} = \frac{-\eta q}{2\pi\rho_0\kappa r}\left\{\left(1-\operatorname{erf}\left(\frac{\xi}{2}\right)\right)-\frac{1}{2\sqrt{\pi}}\int_0^{\xi}\frac{s^2}{\xi^2}e^{-s^2/4}\,ds\right\}.$$

It is useful to write the cartesian stress tensor form of eqn (39a), as a typical example of a spherically symmetric field but also for the purpose of deriving the source dipole solution which is considered next; the result is

$$\sigma_{ij} = \frac{-\eta q}{2\pi\rho_0\kappa r}\left\{\left(1-\operatorname{erf}\left(\frac{\xi}{2}\right)-\frac{1}{2\sqrt{\pi}}\int_0^{\xi}\frac{s^2}{\xi^2}e^{-s^2/4}\,ds\right)\delta_{ij}+\frac{x_ix_j}{r^2}\left(\frac{3}{2\sqrt{\pi}}\int_0^{\xi}\frac{s^2}{\xi^2}e^{-s^2/4}\,ds\right)\right\}. \tag{39b}$$

Adjacent source and sink, a fluid dipole. We mentioned, on inspecting eqn (14a), that both fluid body-forces $\rho_0 f_k^F$ and effective total body-forces F_k (eqn 12) will be responsible for generating a distribution of fluid dipoles: it is convenient, then, to have the solution for a single dipole. Toward this, consider a source of strength $f(t)$ at $n_k\Delta x$ and an equal sink at the origin (where n_k is a directional unit vector): suppose we let Δx become vanishingly small but retain a constant value of $\Omega \equiv f(t)\Delta x$ (the vortex strength). We have then a standard problem of finding the difference between the influence fields of the source and the sink, with the same strength $\Omega/\Delta x$ in the limit as the distance Δx between them vanishes. This problem will recur in considering the point slip and point dilation and there we will be concerned with the difference between influence fields of equal and opposite point forces, of magnitude $T/\Delta x$, with points of application $n_k\Delta x$ apart. Thus we record the general solution here (using non-committal dipole strength P in place of whatever constant is involved, for instance T or Ω). The limiting process is obviously one of differentiating the original field with respect to alterations, in the n_k direction, of the source (or point-force) location; thus, if σ is the field variable of interest and $\sigma^U(x_i, t)$ is its value for a unit source (or force) at the origin, it is easy to show that

$$\sigma^P(x_i, t) = -Pn_k\frac{\partial\sigma^U(x_i, t)}{\partial x_k} \tag{40}$$

is the response to a dipole of strength P at the origin. Formula (40) is valid, of course, only if σ varies purely in magnitude (but not in orientation or any other sense) as the source location is moved while holding the point of evaluation fixed: pore-pressure p, and any component of the cartesian stress tensor, fits into this category.

First, suppose we are interested in an adjacent instantaneous source and sink, $f(t) = Q^{eff}\delta(t)$; the pore-pressure of the resulting dipole, $\Omega \equiv \Omega^{eff}\delta(t)$, may be computed from eqn (36) by means of eqn (40) with $P \equiv \Omega^{eff}$ and the result is

$$p = \frac{\Omega^{eff}c}{8\pi\rho_0\kappa}\left(\frac{n_kx_k}{2\sqrt{\pi}r^5}\right)\xi^5 e^{-\xi^2/4}. \tag{41}$$

A similar operation may be performed on the cartesian stress tensor (analogous to eqn (39b) which results from eqn (37)) but that is not important here, so we leave it out (it is similar to eqn (43) except for time-evolution functions).

The more interesting results appear when $f(t) = qH(t)$, namely that a constant dipole strength, $\Omega \equiv \Omega H(t)$, generates the pore-pressure

$$p = \frac{\Omega}{4\pi\rho_0\kappa}\left(\frac{n_kx_k}{2\sqrt{\pi}r^3}\right)\int_{\xi}^{\infty}s^2e^{-s^2/4}\,ds \tag{42}$$

obtained by using eqn (38) in eqn (40), with $P \equiv \Omega$.

This result shows a remarkable similarity to the pore-pressure field of the point force (eqn 18 and eqn 31) in the n_k direction: precisely, it is the difference between a steady-state harmonic pore-pressure distribution (achieved by the dipole at $t = \infty$) and the pore-pressure due to a point force i.e. it is the point-force solution inverted in time. This bears out perfectly the roles of f_k^F (which generates pure dipoles, no net point forces) and F_k in eqn (14a). It is worth noting distinctions between the dipole and point force, however, especially in the stress-field, which may be derived from eqn (39b),

$$\sigma_{ij} = \frac{\eta\Omega}{\rho_0\kappa}\left[\left(\frac{n_k x_k}{4\pi r^3}\right)\left\{\left(\frac{1}{\sqrt{\pi}}\int_0^\xi\left(1+\frac{3}{\xi^2}\right)s^2 e^{-s^2/4}\,ds - \frac{\xi}{\sqrt{\pi}}e^{-\xi^2/4} - 2\right)\delta_{ij} - \frac{1}{\sqrt{\pi}}\left(15\int_0^\xi \frac{s^2}{\xi^2}e^{-s^2/4}\,ds - 3\xi e^{-\xi^2/4}\right)\frac{x_i x_j}{r^2}\right\} + \frac{3(n_i x_j + n_j x_i)}{4\pi\sqrt{\pi}r^3}\int_0^\xi \frac{s^2}{\xi^2}e^{-s^2/4}\,ds\right]. \tag{43}$$

This dipole stress-field has the same structure as that (eqn 20) for a point force in the n_k direction, but the evolution functions are quite different: for instance, they all vanish at $t = 0$ and the stress-state becomes hydrostatic at $t = \infty$, a quite different behaviour from that in eqn (31).

SIMULATION OF AN ANOMALOUS PLASTIC REGION

Now that the fundamental solutions for a point force or fluid source are available to us, we can actually employ a fairly simple procedure to model fluid injection or zones of plasticity in an otherwise elastic fluid-saturated medium. A general schematic of such an embedded zone is shown in Fig. 3; we consider a bounded body of volume V, surface S (which will be specialised as unbounded later, for actual computations), containing an anomalous region (V^A, surface S^A) in which some distribution of plastic deformation (ϵ_{ij}^p, Δv^p) has developed. The region V^A may also be one into which fluid is being injected (or extracted) through a distribution of "boreholes". We wish to determine the field of influence of this anomalous zone.

The modelling procedure bears a strong resemblance to that frequently employed in simulating inclusions and inhomogeneities in elastic bodies, (e.g. see Eshelby[24]). We imagine that we can remove the region V^A before plastic distortion (ϵ_{ij}^p, Δv^p) takes place, thereby leaving S_+^A stress-free for the moment (or under whatever tractions and pore-pressure it

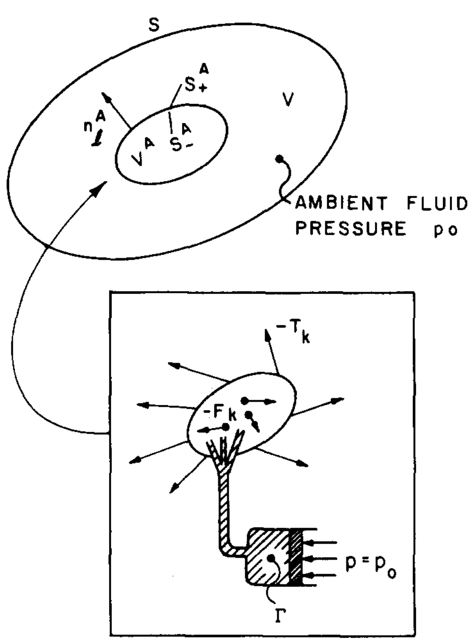

Fig. 3. Schematic for simulating an anomalous zone in a fluid-saturated porous medium. The zone V^A is removed before undergoing the plastic distortion, its original shape is then restored by supplying body-forces and fluid which are withdrawn after it is re-inserted in the body.

endures before the plasticity is induced). We now allow the inelastic deformation ($\epsilon_{ij}^p, \Delta v^p$) to develop in V^A, giving it a distortion which makes it a misfit for its original location in V; further, from a reservoir at zero (or reference) pressure we supply the fluid distribution ($\rho_0 \Delta v^p$) needed to fill the plastically increased pore-space. The next step is to fit this plastically distorted and dilated material, which has as yet undergone no traction changes on $S_-{}^A$ and no pore-pressure changes in its interior, back into its original site in V.

It is suggested by eqn (9), and is easy to prove, that we can restore V^A to its original state of deformation (before ϵ_{ij}^p, Δv^p were allowed to develop) by applying a distribution of body-forces $F_k = (\mathscr{L}_{ijkl}\epsilon_{ij}^p)_{,l}$ over the region V^A and equilibrating them with tractions $T_k = -\mathscr{L}_{ijkl}\epsilon_{ij}^p n_l{}^A$ (where $n_l{}^A$ are components of the normal) on the surface $S_-{}^A$, provided we simultaneously *withdraw* a volume of fluid $\mathscr{L}_{ijkl}B_{kl}\epsilon_{ij}^p$, per unit volume, at each point of V^A. The fluid withdrawal is done at zero (or reference) pressure so the storage reservoir now contains a superfluous mass of fluid $\rho_0(\mathscr{L}_{ijkl}B_{kl}\epsilon_{ij}^p - \Delta v^p) = \Gamma$ which must be put back into each unit volume of V^A at the point where it was extracted.

The region V^A can now be fitted comfortably back into its place in V so that $S_+{}^A$ and $S_-{}^A$ just touch. The last step is to simultaneously bond $S_+{}^A$ to $S_-{}^A$ while relaxing the externally supplied tractions on $S_-{}^A$, which relaxation generates a layer of embedded body-forces $-T_k = \mathscr{L}_{ijkl}\epsilon_{ij}^p n_l{}^A$ on the now interior surface S^A in the volume V. At the same moment we relax the externally supplied body-forces on the interior of V^A, thereby inducing a body-force distribution $-F_k = (-\mathscr{L}_{ijkl}\epsilon_{ij}^p)_{,l}$ over the anomalous (V^A) region of the now integral body V. Lastly, we must (all at the same instant) put back the fluid in the storage reservoir and this is done by introducing a density of instantaneous fluid mass *sources* of strength $\Gamma = \rho_0(\mathscr{L}_{ijkl}B_{kl}\epsilon_{ij}^p - \Delta v^p)$, determined by the plasticity distribution in V^A.

We have thus established that the plasticity in V^A can be modelled by a distribution of body-forces $-F_k$ spread over V^A and a layer of body-forces, of density $-T_k$ per unit surface area of S^A, together with a distribution of instantaneous sources of strength Γ. We presented our arguments for the sudden (but quasi-static) occurrence of the plasticity but any time-dependent plastic distortion can be simulated by a *Duhamel* time integration with the solution for a Heaviside time-dependence. The fluid injection (through "boreholes") has a direct representation as a specified rate or amount of fluid supply i.e. as a spatial and temporal distribution of sources, of the kind leading to either eqns (36), (37) or to eqns (38), (39).

DILATION AND SLIP AS POINT FORCE DIPOLES

The process for simulating an anomalous plastic region (Fig. 3) may be simplified further by establishing the Green functions for plastic strain itself: in the isotropic case there are just two independent influence functions, that for a local (hydrostatic) dilation and that for localised pure shear. We shall obtain both of them by considering the special case where V^A undergoes uniform plastic distortion so that $F_k = 0$; we then simply shrink V^A to a point while retaining a constant value of $\epsilon_{ij}^p V^A$. Since Δv^p and $\mathscr{L}_{ijkl}B_{kl}\epsilon_{ij}^p$ both contribute to an instantaneous source, for which we know the solution (eqns 36 and 37), we already have the part of the influence function which arises from Γ in Fig. 3 so we need only to determine the response to the tractions $T_k = \mathscr{L}_{ijkl}\epsilon_{ij}^p n_l{}^A$.

Point dilation. First consider the situation (Fig. 4) when V^A is a cube which has undergone a uniform plastic dilation $\epsilon_{ij}^p = \frac{1}{3}\epsilon^p \delta_{ij}$; actually, any reasonably equiaxed region will do since it loses all characteristic dimensions when shrunk to a point, but the cube is chosen for ease of computation (similar remarks apply later to plastic slip). For isotropic $\mathscr{L}_{ijkl}$ and B_{kl} (eqns 10), the appropriate uniform strengths of the body-force surface layer T_k and of the source strength Γ (Fig. 4a) are

$$T_k = K\epsilon^p n_k{}^A, \quad \Gamma = \frac{3\rho_0(\nu_u - \nu)\epsilon^p}{B(1-2\nu)(1+\nu_u)}. \tag{44}$$

Plastic straining may be accompanied by pore-space dilation Δv^p; without any loss of generality, we assume (discussion in Chap. 2 of [1]) that Δv^p is related to ϵ_{kk}^p by some microstructural parameter β, $\Delta v^p = \beta\epsilon_{kk}^p$ (typically $\beta \simeq 1.0$). Then, when V^A is shrunk to a point while retaining a constant value of the dilation centre strength $Z = \epsilon^p V^A$, the density of fluid

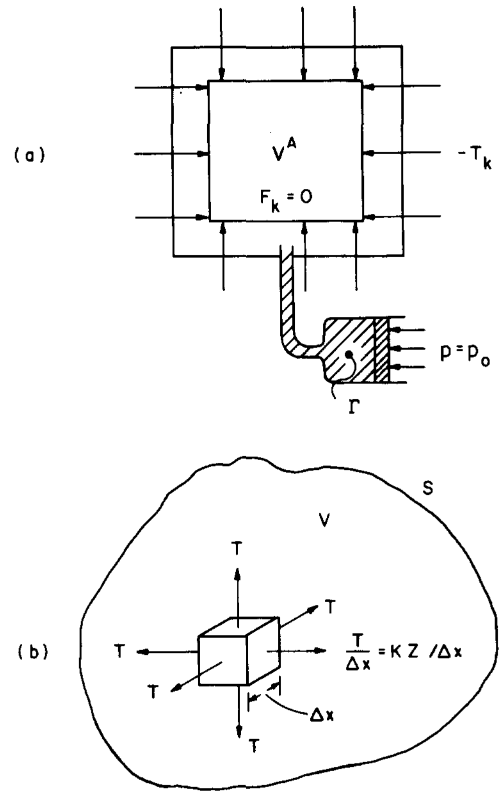

Fig. 4. (a) Cubical anomalous zone undergoing uniform dilation, ϵ^p_{kk}. (b) Zone is shrunk to a point and a triplicate of point-force dipoles is obtained; Z is the limiting finite value of $\epsilon^p_{kk}V^A$ as $V^A \to 0$.

sources $\Gamma - \rho_0 \Delta v^p$ reduces to a single instantaneous source, with response given by eqns (36) and (37) and strength

$$Q^{eff} = \frac{3\rho_0(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)} Z - \rho_0 \beta Z. \tag{45}$$

The solution which we must find, to superpose on this, is that for a centre of compression i.e. a triplicate of dipole pairs of equal and opposite point forces (Fig. 4b) to which the layer of body-forces T_k reduces when V^A is shrunk to a point. The forces have such a magnitude, $KZ/\Delta x$, that their fields do not cancel even when they are brought arbitrarily close, $\Delta x \to 0$. The problem is exactly the same as that leading to eqn (40) and will arise again in the point-slip solution so we use eqn (40) in the most general fashion: suppose a pair of point forces of magnitude $T/\Delta x$ and direction m_k (unit vector) are separated by the vector $n_l \Delta x$ and we allow $\Delta x \to 0$. Then eqn (40), with $P \equiv T$, and eqn (20), with unit magnitude $P_k = m_k$, may be used to get the resulting stress-field of the point-force dipole,

$$\sigma_{ij} = \frac{T}{r^3}\left\{\frac{x_i x_j}{r^2}\left[\frac{n_l x_l m_k x_k}{r^2}(5F_1 - F_1') - n_k m_k F_1\right] - \frac{1}{r^2}[m_k x_k(x_i n_j + x_j n_i)F_1 - n_l x_l(x_i m_j + x_j m_i)(3F_2 - \xi F_2')]\right.$$

$$\left. + n_l x_l m_k x_k (3F_3 - \xi F_3')\frac{\delta_{ij}}{r^2} - n_k m_k F_3 \delta_{ij} - (n_i m_j + n_j m_i)F_2\right\} \tag{46}$$

while eqn (18) yields the pore-pressure field of the dipole

$$p = \frac{T}{r^5}[n_l x_l m_k x_k (3f_1 - \xi f_1') - n_k m_k r^2 f_1]. \tag{47}$$

In the case of the compression centre $n_k = m_k$ for each of the triplicate of dipoles so, for each dipole (which we denote by E for extensional),

$$\sigma_{ij}^E = \frac{T}{r^3}\left[\frac{(m_kx_k)^2}{r^2}\frac{x_ix_j}{r^2}(5F_1 - \xi F_1') - \frac{x_ix_j}{r^2}F_1 - (m_ix_j + m_jx_i)\frac{x_km_k}{r^2}(F_1 + \xi F_2 - 3F_2') - 2m_im_jF_2 + \frac{(m_kx_k)^2}{r^2}(3F_3 - \xi F_3')\delta_{ij} - F_3\delta_{ij}\right] \tag{48a}$$

$$p^E = \frac{T}{r^5}[(m_kx_k)^2(3f_1 - \xi f_1') - r^2f_1]. \tag{48b}$$

The solution in eqns (48) is is useful in its own right as the description of a local transformation which is pure plastic extension† in the direction m_k. To obtain the dilation solution we must now sum over any three orthogonal choices for the vector m_k (e.g. choose it successively along the x_1, x_2 and x_3 axes) and the final result is

$$\sigma_{ij}^Z = \frac{KZ}{r^3}\left[\frac{x_ix_j}{r^2}(6F_2 - \xi F_1' - 2\xi F_2') - \delta_{ij}(2F_2 + \xi F_3)\right] \tag{49a}$$

$$p^Z = \frac{-KZ}{r^3}\xi f_1'(\xi). \tag{49b}$$

The first observation is that the field is spherically symmetric (e.g. see eqn (39)) and thus can be recorded in the notation of eqns (33)–(35); in so doing we apply eqn (31) to F_1, F_2, F_3, f_1 and obtain

$$\sigma_{rr}^Z = \frac{-KZ}{2\pi(1-\nu)r^3}\left[(1-2\nu) - \left(\frac{\nu_u - \nu}{1-\nu_u}\right)\frac{1}{2\sqrt{\pi}}\int_0^\xi \eta^2 e^{-\eta^2/4}\,d\eta\right] \tag{50a}$$

$$\sigma_{\theta\theta}^Z = \frac{KZ}{4\pi(1-\nu)r^3}\left[(1-2\nu) - \left(\frac{\nu_u - \nu}{1-\nu_u}\right)\frac{1}{2\sqrt{\pi}}\left\{\int_0^\xi \eta^2 e^{-\eta^2/4}\,d\eta - \xi^3 e^{-\xi^2/4}\right\}\right] \tag{50b}$$

$$p^Z = \frac{-KZ}{4\pi r^3}\left[\frac{B(1+\nu_u)}{3(1-\nu_u)2\sqrt{\pi}}\xi^3 e^{-\xi^2/4}\right]. \tag{50c}$$

The second observation is that eqn (50c) has the same form as eqn (36) for the instantaneous source, an extremely convenient idea in solving for a plastically dilating region, represented by a distribution of sources and point dilations (e.g. in the dilating sphere solution described in [1]). Even the stresses in eqns (50a) and (50b) are identical to those in eqn (37), except for a residual elastic term, $KZ(1-2\nu)/2\pi(1-\nu)r^3$, which remains at $t = \infty$ and is actually purely additive to the solution (eqns (36) and (37)) for an instantaneous point source of a strength given by

$$Q^{eff} = -\frac{\rho_0\kappa B(1+\nu_u)}{3c(1-\nu_u)}KZ. \tag{50d}$$

This kind of correspondence between the field of a point dilation (eqns 50) and that of a source (eqns 36 and 37) is a useful tool and has been exploited in our solution for the dilating sphere (Chap. 3 of [1]).

These solutions for dilation and fluid injection have substantial importance in such problems as underground fracturing of saturated rocks and in earthquake-related occurrence of dilatant anomalous zones (e.g. [25]) but they must usually be coupled to the occurence of plastic shearing and the realistic simulation of actual problems is a separate matter (Chap. 4 of [1]). Our only purpose here is to provide the fundamental solutions for such physically reasonable anomalies in saturated rocks, soils and other porous media.

†Obviously, one needs to make a small adjustment for Poisson contraction in the restoration process so solutions for accompanying dipoles of strength νT, in orthogonal directions, must be added to eqn (48) to simulate pure tensile cracking.

Pure shear at a point. The second influence function for plastic strain is generated by having the anomalous region V^A (Fig. 3) undergo a uniform pure shear $\epsilon_{ij}^p = \epsilon_{MN}^p(m_i n_j + m_j n_i)$, $\Delta v^p = 0$, perhaps as a result of simultaneous slip on planes with orthogonal normals m_k and n_k; V^A may be any reasonably equiaxed parallelipiped but we choose a cube again (Fig. 5a) for ease of computation. Body-forces F_k vanish and, when $\mathscr{L}_{ijkl}$ and B_{kl} are isotropic (eqns 10), Γ disappears also. The restoration of V^A to its original state can thus be accomplished by a surface traction distribution, given by

$$-T_k = -\mathscr{L}_{ijkl}\epsilon_{ij}^p n_l^A = -2G\epsilon_{MN}^p(m_k n_l + m_l n_k)n_l^A. \tag{51}$$

Now V^A is shrunk to a point (or viewed from a sufficiently large distance) while retaining a constant magnitude of the product $V^A\epsilon_{MN}^p$; if Δx is a typical side the tractions must thus reduce to a pair of force dipoles (Fig. 5b) where each point force has a strength $2GM/\Delta x$ and Δx is vanishingly small. The quantity $2GM$ (analogous to KZ used for the dilation strength earlier) is often termed the "moment" (e.g. by Nur[26]) of the slip and is the limit of $2G\epsilon_{MN}^p V^A$ as V^A shrinks to a point.

The point slip solution is thus the same as that for a momentless pair of point-force dipoles (Fig. 5b) with strength $2GM$, so we can employ eqn (47) to obtain (by the same logic as that leading to eqn 40)

$$\begin{aligned}\sigma_{ij} &= -2GM[n_l\sigma_{ij,l}^{(m_k)} + m_l\sigma_{ij,l}^{(n_k)}]\\ &= \frac{2GM}{r^5}\Big[(n_lx_l(x_im_j + x_jm_i) + m_kx_k(x_in_j + x_jn_i))(3F_2 - \xi F_2' - F_2) + \frac{2}{r^2}(n_lx_l)(m_kx_k)x_ix_j(5F_1 - \xi F_1')\\ &\quad - 2(n_im_j + m_jn_i)r^2F_2 + 2(n_lx_l)(m_kx_k)(3F_3 - \xi F_3')\delta_{ij}\Big]\end{aligned} \tag{52a}$$

$$p = \frac{4GM}{r^5}(n_lx_l)(m_kx_k)(3f_1 - \xi f_1') \tag{52b}$$

where we have employed the notation $\sigma_{ij}^{(m_k)}$ for the stress-field of a unit point-force in the m_k direction at the origin of co-ordinates. The solution in eqn (52) is all that we formally require to establish the stress-field of any plastically sheared region or slip-surface. The computation, by superposing a distribution of point slippages, would normally be carried out numerically but we must mention a useful example, the circular planar slipped region (or dislocation loop), where

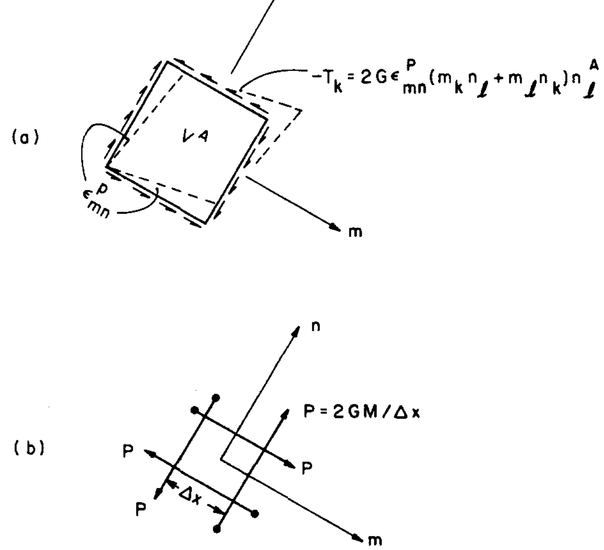

Fig. 5. (a) Cubical anomalous zone undergoing uniform pure shear, ϵ_{mn}^P. (b) Zone is shrunk to a point and we obtain a momentless pair of point-force dipoles; M is the limiting value of $\epsilon_{mn}^P V^A$.

analytical features can be extracted (Chap. 3 of [1]). Some discussion of both analytical and numerical implementation, with the array of fundamental solutions now available to us, seems essential, so applications are discussed next.

APPLICATIONS AND CONCLUSIONS

Since the modelling of an anomalous region (Fig. 3) was independent of boundary S, the occurrence of distributed plasticity ($\epsilon_{ij}^{p}, \Delta v^{p}$) and fluid injection ($R^{F}$) in any region may be simulated by an appropriate density of point slip (Fig. 5), point dilation (Fig. 4) and fluid sources: the influence functions in eqns (52), (50) and (39) may be used provided that the boundary conditions can be satisfied.

In the case of an infinite region V there is no need to modify these influence functions and any specified plasticity distribution has a readily computable "consolidation" response: for instance, we have used (Chap. 3 of [1]) a dilation distribution over a spherical region to examine the effect of anomalous dilatancy in earthquake vicinities [25] and an inelastic shear distribution over a plane circular surface to analytically determine the evolution of the stress field around a bounded earth fault after the occurrence of a sudden slip (e.g. leading to aftershocks and secondary slippage [26]). Obviously, if one is interested only in the far-field of such anomalies, then there is no need to assign them a characteristic shape and the influence functions themselves suffice to evaluate the time-dependent stress field.

If the region V is not infinite, the modelling, in essence, involves distributing a further density of point forces and fluid sources (with influence given by eqns (18), (20), (31), (38) and (39)) along the desired locus of S in such a fashion that their effect just makes up the difference between the embedded plasticity field and the desired boundary-conditions (e.g. specified $\sigma_{ij}n_i$ and p) on S. Both a direct implementation of this notion and a more natural version (in Laplace-transformed time variable) following from the reciprocal theorem of eqn (9), have been described in Chap. 4 of [1]: the latter is somewhat more analogous to conventional numerical boundary integral equation schemes (e.g. [27]) but it has less flexibility than the direct method, especially for simulation of propagating cracks and shear faults.

In fact, a major motivation for establishing the fundamental solutions was the desire to model just such time-dependent fracture propagation problems (e.g. as reviewed in Chap. 1 of [1]). The concept is that progressive sliding (or opening) on a rupture surface can be simulated as a density of concentrated slip (or extension) increasing just sufficiently to preserve some (e.g. frictional) stress criterion on the surface. Such modelling has been conducted for the limiting elastic contexts (e.g. [28]) and seems presently to be the only tractable procedure for modelling progressive formation of displacement discontinuities. Thus, apart from their inherent value as the first basic set of fundamental solutions for coupled linearized deformation-diffusion, the influence functions have extensive applications [1] in understanding many geophysical, geotechnical and (apparently) glaciological phenomena.

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APPENDIX

(a) *Proof of reciprocal theorem.* We begin with the obvious expression of the reciprocity in eqns (3), namely that

$$\tilde{\sigma}_{ij}^{(1)}[\tilde{\epsilon}_{ij}^{(2)}-\tilde{\epsilon}_{ij}^{p(2)}]-\tilde{\sigma}_{ij}^{(2)}[\tilde{\epsilon}_{ij}^{(1)}-\tilde{\epsilon}_{ij}^{p(1)}]+\tilde{p}^{(1)}[\Delta\tilde{v}^{(2)}-\Delta\tilde{v}^{p(2)}]-\tilde{p}^{(2)}[\Delta\tilde{v}^{(1)}-\Delta\tilde{v}^{p(1)}]=0 \tag{A1}$$

in which we are using tilde to denote the Laplace transform (eqn 9c). We recall eqns (1), (3) and (4) and eqn (A1) then becomes

$$\begin{aligned}&[\tilde{u}_i^{(2)}\tilde{\sigma}_{ij}^{(1)}]_{,j}-[\tilde{u}_i^{(1)}\tilde{\sigma}_{ij}^{(2)}]_{,j}-\tilde{u}_i^{(2)}\tilde{\sigma}_{ij,j}^{(1)}+\tilde{u}_i^{(1)}\tilde{\sigma}_{ij,j}^{(2)}+\mathscr{L}_{ijkl}B_{kl}[\tilde{p}^{(1)}\tilde{\epsilon}_{ij}^{p(2)}-\tilde{p}^{(2)}\tilde{\epsilon}_{ij}^{p(1)}]+(\mathscr{L}_{ijkl}\tilde{\epsilon}_{ij}^{p(2)})_{,l}\tilde{u}_k^{(1)}\\&\quad-(\mathscr{L}_{ijkl}\tilde{\epsilon}_{ij}^{p(1)})_{,l}\tilde{u}_k^{(2)}-(\mathscr{L}_{ijkl}\tilde{u}_k^{(1)}\tilde{\epsilon}_{ij}^{p(2)}-\mathscr{L}_{ijkl}\tilde{u}_k^{(2)}\tilde{\epsilon}_{ij}^{p(1)})_{,l}-\tilde{p}^{(1)}\Delta\tilde{v}^{p(2)}+\tilde{p}^{(2)}\Delta\tilde{v}^{p(1)}+[\tilde{p}^{(1)}\Delta\tilde{m}^{(2)}-\tilde{p}^{(2)}\Delta\tilde{m}^{(1)}]/\rho_0=0\end{aligned} \tag{A2}$$

We now employ the two transformed eqns (8) and (5), namely

$$\Delta\tilde{m}/\rho_0=-\tilde{Q}_{i,i}+\tilde{R}^F,\quad \tilde{Q}_i=-s^{-1}\kappa(\tilde{p}_{,i}-\rho_0\tilde{f}_i^F) \tag{A3}$$

and these imply that

$$\begin{aligned}(\tilde{p}^{(1)}\Delta\tilde{m}^{(2)}-\tilde{p}^{(2)}\Delta\tilde{m}^{(1)})/\rho_0-(\tilde{p}^{(1)}\tilde{R}^{F(2)}-\tilde{p}^{(2)}\tilde{R}^{F(1)})+(\tilde{p}^{(1)}\tilde{Q}_i^{(2)}-\tilde{p}^{(2)}\tilde{Q}_i^{(1)})_{,i}&=\tilde{p}_{,i}^{(1)}\tilde{Q}_i^{(2)}-\tilde{p}_{,i}^{(2)}\tilde{Q}_i^{(1)}\\&=\rho_0[\tilde{f}_i^{F(1)}\tilde{Q}_i^{(2)}-\tilde{f}_i^{F(2)}\tilde{Q}_i^{(1)}].\end{aligned} \tag{A4}$$

We now integrate eqn (A2) over the region V with surface S and employ the divergence theorem; remembering eqn (6), the result is

$$\begin{aligned}&\int_S[\tilde{\sigma}_{ij}^{(1)}\tilde{u}_i^{(2)}-\tilde{\sigma}_{ij}^{(2)}\tilde{u}_i^{(1)}-\tilde{p}^{(1)}\tilde{Q}_j^{(2)}+\tilde{p}^{(2)}\tilde{Q}_j^{(1)}+\mathscr{L}_{ijkl}(\tilde{\epsilon}_{kl}^{p(1)}\tilde{u}_i^{(2)}-\tilde{\epsilon}_{ij}^{p(2)}\tilde{u}_i^{(1)})]n_j\,\mathrm{d}S\\&\quad+\int_V[(\tilde{f}_k^{(1)}-(\mathscr{L}_{ijkl}\tilde{\epsilon}_{ij}^{p(1)})_{,l})\tilde{u}_k^{(2)}-(\tilde{f}_k^{(2)}-(\mathscr{L}_{ijkl}\tilde{\epsilon}_{ij}^{p(2)})_{,l})\tilde{u}_k^{(1)}]\,\mathrm{d}V\\&\quad+\int_V\{\rho_0(\tilde{f}_i^{F(1)}\tilde{Q}_i^{(2)}-\tilde{f}_i^{F(2)}\tilde{Q}_i^{(1)})+\tilde{p}^{(1)}[\tilde{R}^{F(2)}-\Delta\tilde{v}^{p(2)}+\mathscr{L}_{ijkl}B_{kl}\tilde{\epsilon}_{ij}^{p(2)}]\}\,\mathrm{d}V\\&\quad-\int_V\tilde{p}^{(2)}[\tilde{R}^{F(1)}-\Delta\tilde{v}^{p(1)}+\mathscr{L}_{ijkl}B_{kl}\tilde{\epsilon}_{ij}^{p(1)}]\,\mathrm{d}V=0.\end{aligned} \tag{A5}$$

With reorganisation of terms, this is the desired result written in eqn (9). We note that the only restrictions on the fields used in the theorem is that they correspond initially ($t=0$) to an undrained deformation; thus the imposition of loading must begin at $t=0$.

(b) *Field equations for an anisotropic medium.* The equilibrium equations for displacement take the very obvious form

$$\mathscr{L}_{ijkl}\left[\frac{1}{2}(u_{k,lj}+u_{l,kj})\right]=f_i+\mathscr{L}_{ijkl}\epsilon^p_{kl,j}+\mathscr{L}_{ijkl}B_{kl}p_{,j} \tag{A6}$$

which shows the gradient of plastic strains in the same role as the total body force f_i. The equation of mass conservation takes the form

$$\kappa_{ij}p_{,ij}=\left(D+\frac{v_0}{K_f}\right)\frac{\partial p}{\partial t}+B_{kl}\frac{\partial\sigma_{kl}}{\partial t}+\rho_0\kappa_{ij}f^F_{j,i}+\frac{\partial v^p}{\partial t}-r^F/\rho_0. \tag{A7}$$

We would like to reduce this to some kind of diffusion equation (actually[4], a "pseudo-parabolic" equation) in a single variable, some combination of p and σ_{kl}. To do this we would need an equation for $B_{kl}\sigma_{kl,ij}$ (by analogy with the isotropic problem); we might expect to obtain such a relation from the compatibility equations but this is not always the case. It seems eqn (A7) is the best we can do with mass conservation: the major benefit is that the plastic porosity change Δv^p and the fluid body forces f_k^F are shown as corresponding to a source term r^F. However, the more significant comment is that there are no well-known solutions of equations like (A7) except in the isotropic limit. (Since k_{ij} is symmetric, it has principal values and the principal co-ordinates can, of course, be scaled to give an isotropic homogeneous form of eqn (A7); physically, we expect B_{ij} to have frequently the same principal directions as k_{ij} and $\mathscr{L}_{ijkl}$ to reveal uncoupled shear and normal stress response in the system of principal co-ordinates. If so, then the simultaneous solution of eqns (A6) and (A7) becomes more tractable). The general problem of establishing anisotropic fundamental solutions thus seems like a very formidable one, given that there is already great difficulty (e.g. see [29]) in solving eqns (A6), even without the coupling to eqn (A7). However, the methodology of using symmetry arguments (as employed in arriving at eqns (19) and (20) for instance), has aided us considerably in surmounting some of the difficulties involved here: for instance, the solution for a transversely isotropic medium is presently being worked out with the motivation of application to stratified geological structures. Given the potential for immediate application to simulation of faulting phenomena (especially in the earthquake context[26] where any strong anisotropy can have profound temporary stabilization effects, by pore-suction induction on the fault surface, whenever pore-fluid is present[1]) we propose the establishment of tractable fundamental anisotropic solutions as an important area of endeavor.